CALCULATION OF THE DYNAMICS OF A FLEXIBLE CIRCULAR PLATE ON A LINEARLY DEFORMABLE BASE

O. D. Pryakhina, V. V. Satarova, and O. M. Tukodova UDC 539.3

In this paper we study the dynamical contact problem of the interaction of a deformable circular plate with radius b with an elastic or viscoelastic semibounded medium, such as a layer, a packet of layers, and a layered half-space. An axisymmetric transverse external load of the form $p_0(r)\exp(-i\omega t)$, where ω is the frequency of the vibrations and t is the time, acts on the plate. The contact between the plate and the base is frictionless. We consider a wide class of conditions of clamping of the edges of the plate: a) sliding contact, b) hinged edge, c) clamped edge, and d) free edge.

The axisymmetric problem is solved by an approach previously developed for solving similar two-dimensional problems [1]. The approach was also found to be effective for axisymmetric problems. The method is based on using the characteristic forms of vibrations of an elastic body of finite size and the method of fictitious absorption, employed for solving integral equations arising in dynamical problems of the theory of elasticity and having strongly oscillating kernels.

Algorithms and programs, which make it possible to analyze the main characteristics of the interaction of circular plates with the base – deflections of the plate and stresses in the plate–base contact region – were developed on the basis of the solutions constructed. Numerical analysis was performed for media consisting of an elastic or viscoelastic layer, rigidly attached to a nondeformable base. The method also makes it possible to analyze more complicated media.

1. Basic Equations. Construction of the Solution. The basic equation describing vibrations of a circular plate, whose stress–strain state is described by the technical theory of bending, has the following form in terms of dimensionless amplitude parameters [the factor $\exp(-i\omega t)$ is dropped everywhere]

$$D_0(\partial^2/\partial r^2 + 1/r\cdot\partial/\partial r)^2 w(r) - RH\Omega^2 w(r) = p(r) - q(r). \tag{1.1}$$

Here $D_0 = H^3/[12M(1-\nu_0^2)]$; $H = h_0/h$; $R = \rho_0/\rho$; $M = \mu/E_0$; $\Omega^2 = \exp(-i\gamma)\rho\omega^2h^2/\mu$; $a = b/h$; $\mu = E/(2(1+\nu))$ is the Lamé constant for the medium; E_0, ν_0, h_0, ρ_0 are, respectively, Young's modulus, the Poisson ratio, and the thickness and density of the material of the plate; E, ν, h, ρ are the analogous characteristics of the medium; γ is the viscosity parameter of the medium (Sorokin's coefficient of losses to internal friction in the material [2]); Ω is the dimensionless frequency of the vibrations; w(r) and a are the deflection and radius of the plate, scaled to the characteristic geometric parameter h of the medium (for example, the thickness of the layer); and $p(r) = p_0(r)$ and q(r) are a prescribed load and the reaction of the base or contact stresses, scaled to μ. This function is determined by solving the standard dynamical contact problem of the action of an absolutely rigid circular stamp on a medium:

$$\int_0^a k(r,\xi)\,q(\xi)\,\xi d\xi = f(r),\ 0 \leqslant r \leqslant a, \qquad k(r,\xi) = \int_\sigma K(\alpha)\,J_0(\alpha r)\,J_0(\alpha\xi)\,\alpha d\alpha \tag{1.2}$$

where the function f(r) describes the shape of the base of the stamp and $J_0(r)$ is the zeroth-order Bessel function.

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The function $K(\alpha)$ is determined by the type of medium. The contour σ is chosen in accordance with the radiation principle [3] and is identical to the real half-space $(0, \infty)$ for a medium with internal friction. For an elastic medium the contour σ circumscribes real zeros z_ℓ and the poles p_ℓ, where $\ell = 1, 2, \ldots, N_0$, of the function $K(\alpha)$ from below, and the rest of the contour coincides with the real half-space. Equations (1.1) and (1.2) must be supplemented by the condition that the displacements be equal in the plate-base contact region

$$w(r) = f(r),\ 0 \leqslant r \leqslant a,\ z = 0$$

and the boundary conditions at the edges of the plate $r = a$ and $z = 0$, which depend on the type of contact:

a) sliding contact

$$\partial w/\partial r = 0 \text{ for } r = a,$$
$$Q(r) = \partial^3 w/\partial r^3 + 1/r \cdot \partial^2 w/\partial r^2 - 1/r^2 \cdot \partial w/\partial r = 0 \text{ for } r = a;$$

b) hinged edge

$$w(a) = 0,\ M(r) = \partial^2 w/\partial r^2 + \nu_0/r \cdot \partial w/\partial r = 0,\ r = a;$$

c) clamped edge

$$w(r) = \partial w/\partial r = 0,\ r = a;$$

d) free edge

$$M(a) = Q(a) = 0$$

where $M(r)$ and $Q(r)$ are the moment and the reduced transverse force.

We seek the solution of Eqs. (1.1) and (1.2) in the form

$$w(r) = \sum_{m=1}^{\infty} A_m \varphi_m(r),\ \varphi_m(r) = [J_0(\theta_m r) - j_m I_0(\theta_m r)], \tag{1.3}$$

where j_m is chosen as follows:

a) $j_m = 0$;

b, c) $j_m = J_0(\theta_m a)/I_0(\theta_m a)$;

d) $j_m = J_1(\theta_m a)/I_1(\theta_m a)$, $\varphi_1 = 1$,

where $I_0(z)$ and $I_1(z)$ are zeroth- and first-order modified Bessel functions.

The unknowns θ_m for the problems a-d are determined from transcendental equations, whose form is determined by the boundary conditions at the edges of the plate $r = a$:

a) $J_1(\theta_m a) = 0$;

b) $L(\theta_m a) = 2\theta_m a(1 - \nu_0)^{-1} I_0(\theta_m a) J_0(\theta_m a)$;

c) $L(\theta_m a) = 0$;

d) $L(\theta_m a) = 2(1 - \nu_0) I_1(\theta_m a) J_1(\theta_m a)$, $L(z) = J_1(z) I_0(z) + I_1(z) J_0(z)$.

The solution $t(r, \eta)$ of the integral equation (1.2) for the right-hand side $w(r) = J_0(\eta r)$ was constructed in [4] by the fictitious-absorption method. Using the results of this work for the displacements $w(r)$ of the type (1.3), we determine the contact stresses by the relations

$$q(r) = \sum_{m=1}^{\infty} A_m q_m(r),\ q_m(r) = [t(r, \theta_m) - j_m t(r, i\theta_m)]. \tag{1.4}$$

A. Far from the edge of the plate the function $t(r, \eta)$ has the form

$$t(r,\eta)=J_0(\eta r)K^{-1}(\eta)+\pi iac^{-1}\Big\{\sqrt{\eta^2+B^2}\sum_{l=1}^{N} z_l \operatorname{Res} H^{-1}(z_l)\times$$
$$\times J_0(z_l r)F_1(\eta, z_l)-\pi i\sum_{k=1}^{N} c_k\sum_{j=1}^{N} p_j \operatorname{Res} H(p_j)\ J_0(p_j r_k)\sum_{l=1}^{N} z_l \operatorname{Res} H^{-1}(z_l)J_0(z_l r)F_2(p_j, z_l)\Big\}.$$

The unknowns c_k are determined from the equations

$$\pi i\sum_{k=1}^{N} c_k\sum_{j=1}^{N} p_j \operatorname{Res} H(p_j)J_0(p_j r_k)F_1(z_l, p_j)=\sqrt{\eta^2+B^2}F_0(z_l,\eta),\ l=1,\ 2,\ \ldots, N.$$

Here

$$F_0(\alpha,\eta)=\begin{cases} a\left[J_1^2(\alpha a)+J_0^2(\alpha a)\right]/2, & \eta=\alpha,\\ [\eta J_0(\alpha a)J_1(\eta a)-\alpha J_1(\alpha a)J_0(\eta a)]/(\eta^2-\alpha^2);\end{cases}$$
$$F_1(\alpha,\eta)=\left[\eta J_0(\alpha a)H_1^{(1)}(\eta a)-\alpha J_1(\alpha a)H_0^{(1)}(\eta a)\right]/(\eta^2-\alpha^2);$$
$$F_2(\alpha,\eta)=\left[\alpha H_1^{(1)}(\alpha a)H_0^{(1)}(\eta a)-\eta H_0^{(1)}(\alpha a)H_1^{(1)}(\eta a)\right]/(\alpha^2-\eta^2);$$
$$H(\alpha)=\sqrt{\alpha^2+B^2}K(\alpha)/c;\ c=\lim_{|\alpha|\to\infty}|\alpha|K(\alpha);$$

$H_n^{(1)}$ is a first-order Hankel function; r_k are the points which partition the interval (0, a) into equal segments; B is an approximation parameter, which is chosen in accordance with the ficticious-absorption method ($B \gg 1$); z_ℓ and p_ℓ are the zeros and poles of $K(\alpha)$, such that $\operatorname{Im} z_\ell$, $p_\ell \geq 0$ for an elastic medium and $\operatorname{Im} z_\ell$, $p_\ell > 0$ for a viscoelastic medium.

B. Near the edge of the plate the function $t(r, \eta)$ assumes the form

$$t(r,\eta)=c^{-1}\Big[C(\eta)/\sqrt{a^2-r^2}+\pi C(\eta)F_3(r)-$$
$$-\pi i/4\sum_{k=1}^{N} c_k\left(a^2F_4(r_k)/\sqrt{a^2-r^2}+\pi a^2F_3(r)F_4(r_k)+4F_5(r,r_k)\right)\Big],$$

where $C(\eta) = a^2\eta[J_1{}^2(\eta a) + J_0{}^2(\eta a)]\sqrt{\eta^2 + B^2}(2\sin(a\eta))^{-1}$;

$$F_3(r)=\sum_{l=1}^{N}\operatorname{Res} H^{-1}(z_l)\exp(iz_l a)J_0(z_l r);$$
$$F_4(r_k)=\sum_{j=1}^{N}2p_j^2\operatorname{Res} H(p_j)J_0(p_j r_k)\left[J_1(p_j a)H_1^{(1)}(p_j a)+\right.$$
$$\left.+J_0(p_j a)H_0^{(1)}(p_j a)\right]/\sin(p_j a);$$
$$F_5(r,r_k)=\sum_{l=1}^{N} z_l\operatorname{Res} H^{-1}(z_l)\begin{cases}H_0^{(1)}(z_l r)J_0(z_l r_k), & r\geqslant r_k,\\ H_0^{(1)}(z_l r_k)J_0(z_l r), & r\leqslant r_k.\end{cases}$$

The unknown coefficients c_k are determined from the equations

$$\sum_{k=1}^{N} c_k\left[z_l J_0(z_l r_k)+\pi i a^2\sin(z_l a)F_4(r_k)/4\right]=C(\eta)\sin(z_l a),\ l=1,2,\ldots,N.$$

Substituting into the differential equation (1.1) the expressions for $w(r)$ and $q(r)$ in the form of Eqs. (1.3) and (1.4), multiplying by $\varphi_n(r)$, and integrating over r from zero to a, we obtain a system of linear algebraic equations for the unknown coefficients A_m:

$$\alpha_n A_n + \sum_{m=1}^{\infty} s_{nm} A_m = p_n, \; n = 1, 2, \ldots, \infty. \tag{1.5}$$

Here $s_{mn} = \int_0^a q_m(r)\varphi_n(r)\,rdr;\; p_n = \int_0^a p(r)\varphi_n(r)\,rdr;\; \alpha_n = \left(D_0\theta_n^4 - RH\Omega^2\right)\Delta_n;\;\; \Delta_n = \int_0^a \varphi_n^2(r)\,rdr.$

The elements of the matrix $S = \|s_{mn}\|$ and the functions Δ_n are calculated by direct integration, taking into account Eq. (1.4), and are analytic functions of the frequency Ω:

$$s_{mn} = T(\theta_n, \theta_m) - j_m T(\theta_n, i\theta_m) - j_n T(i\theta_n, \theta_m) + \\ + j_n j_m T(i\theta_n, i\theta_m).$$

The resultant of the contact pressures or the reaction of the base is determined by the formula

$$Q = \int_0^a q(r)\,rdr = \sum_{k=1}^{\infty} A_k [T(0, \theta_k) - j_k T(0, i\theta_k)]. \tag{1.6}$$

The function $T(\alpha, \eta)$ is the Bessel transform of the function $t(r, \eta)$ and far from the edge of the plate it has the form

$$T(\alpha, \eta) = a\left\{F_0(\alpha, \eta)\sqrt{\eta^2 + B^2} - \pi i \sum_{k=1}^{N} c_k \sum_{j=1}^{N} p_j \operatorname{Res} H(p_j) \;\; J_0(p_j r_k) F_1(\alpha, p_j)\right\} \Big/ c/H(\alpha),$$

while near the edge of the plate

$$T(\alpha, \eta) = \left\{C(\eta)\sin(\alpha a)/\alpha - \sum_{k=1}^{N} c_k [J_0(\alpha r_k) + \pi i a^2 \sin(\alpha a) F_4(r_k)/4/\alpha]\right\} \Big/ c/H(\alpha).$$

In order to construct an approximate solution $w(r)$ and $q(r)$ it is sufficient to retain in Eqs. (1.3)-(1.5) only several terms of the series in order to achieve prescribed accuracy, since the coefficients A_m approach zero rapidly as the parameter m increases.

2. Numerical Analysis of the Solution. The numerical analysis was performed for elastic and viscoelastic layer, rigidly attached to a nondeformable base, and two types of loads: $p(r) = 1$ and $p(r) = \delta(r)$, where $\delta(r)$ is the Dirac delta function. The layer occupies the region $-\infty \le x, y \le \infty$, $0 \le z \le h$.

We studied the effect of the conditions of clamping of the plate, the type of loading, the viscosity of the medium, and the rigidity and geometric dimensions of the plate on the distribution of the amplitudes of the deflections $w(r)$, the contact pressures $q(r)$, and total forces Q in the region of contact as a function of the reduced frequency Ω.

For the indicated medium the function $K(\alpha)$ has the form presented in [3, 4]. For numerical implementation, the real and complex zeros and the poles $K(\alpha)$ are calculated first, and then the approximating function

$$K_*(\alpha) = c(\alpha^2 + B^2)^{-1/2} \prod_{k=1}^{N} (\alpha^2 - z_k^2)(\alpha^2 - p_k^2)^{-1}$$

is constructed, such that $|K(\alpha) - K_*(\alpha)| < \varepsilon$ for any prescribed number $\varepsilon > 0$. According to [3], this also guarantees that the solutions of the integral equations $K_q = w$ and $K_* q = w$ are close. Next, the amplitude-frequency characteristics of the problem are calculated from the formulas (1.3), (1.4), and (1.6).

Figure 1 illustrates the effect of the rigidity of the plate $M = \mu/E_0$ for sliding contact at the edge on the distribution of the amplitude of the deflections Re $w(r)$ when a uniformly distributed load of unit amplitude $p(r) = 1$ acts on the plate. The curves 1-3 corre-

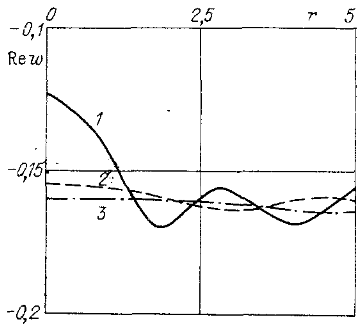

Fig. 1

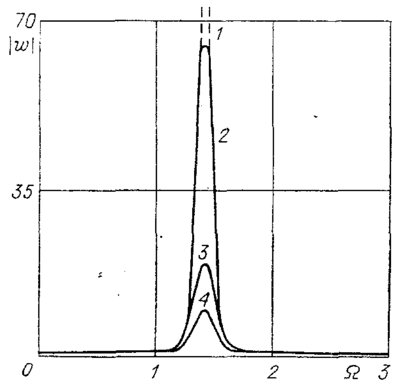

Fig. 2

spond to M = 0.4, 0.04, and 0.004. The remaining parameters are: R = 1.25, H = 0.8, a = 5, Ω = 2.6, ν = 0.3, ν_0 = 0.3, and γ = 0.

Analysis shows that when the rigidity of the plate E_0 decreases (i.e., the parameter M increases with fixed rigidity of the medium), the amplitudes w(r) and q(r) and their oscillations increase, and the more rigid the plate, the more uniform the distribution of deflections and contact pressures is.

Similar results can be obtained by studying the effect of the geometric parameters of the plate on the behavior of w(r) and q(r) for specific values of Ω. As the radius of the plate increases, the oscillations increase, and as the thickness of the plate increases, the vibrations decrease. These relations are more pronounced at frequencies above the critical frequency of the waveguide in the elastic medium Ω_* (for a layer $\Omega_* = \pi/2$). At frequencies $\Omega \leq \Omega_*$ a change in the elastic and geometric parameters can have virtually no effect on the amplitude of the contact stresses and deflections.

The effect of the viscosity γ of the base (Fig. 2) indicates that the internal friction significantly affects the resonances of the system. As γ increases, the amplitude of the vibrations decreases. The curves 1-4 correspond to γ = 0, 0.01, 0.1, and 0.2, where p(r) = 1, a = 7, M = 0.004, R = 1.25, H = 0.8, ν = 0.3, ν_0 = 0.3, and r = 0.

The difference in the maximum amplitudes of the vibrations for different values of γ confirms that the internal friction in the material of the base must be taken into account in the dynamical calculation of foundations and structures as a whole.

3. Resonances of the System. Figure 3 illustrates the effect of the relative thickness of the plate $H = h_0/h$ on the resonances of the system consisting of a circular plate and an elastic layer. The curves 1-4 correspond to H = 0.5, 1.5, 2.5, and 4 with p(r) = 1, a = 11.5, M = 0.0004, R = 1.25, ν = 0.3, ν_0 = 0.3, γ = 0, and r = 0. It is obvious that as the parameter H increases, the resonance frequency decreases and the peak becomes narrower. In the region $\Omega > \Omega_*$, where Ω_* is the critical frequency of excitation of waves and $\Omega_* = \pi/2$ in a layer, resonance of the system is bounded, and for $\Omega \leq \Omega_*$ and starting at some ratio of the geometric and elastic parameters of the plate—elastic base system there appear so-called B resonances, at which the amplitude of the vibrations grows without bound. These resonances for a system consisting of an absolutely stiff stamp or beam — elastic strip were studied in [1, 5], and they have real values for $\Omega \leq \Omega_*$ and complex values for $\Omega > \Omega_*$. For this reason, in Fig. 3 for $\Omega > \pi/2$ and real values of the frequencies the growth of the amplitude of the vibrations is bounded.

For the plate—viscoelastic layer system the amplitudes of the main characteristics will grow without bound only for complex values of the vibrational frequencies, so-called characteristic frequencies. For a viscoelastic base, to each characteristic form of the vibrations there is associated a complex characteristic frequency. The resonance frequencies are determined by the vanishing of the determinant of the system (1.5):

$$\Delta(\Omega) = \det\|\alpha_l\delta_{kl} + s_{kl}\|_{k,l} = 0$$

where $\delta_{k\ell}$ is the Kronecker delta function.

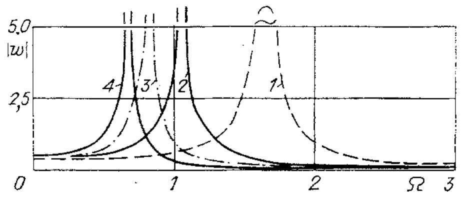

Fig. 3

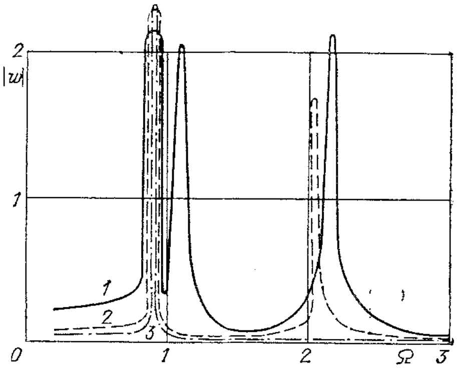

Fig. 4

The method employed in this work for searching for the complex roots of this equation depends on the "principle of the argument" well known in theory and, significantly, there is no need to have initial approximations in order to search for the roots and it is also not necessary to have any information about the multiplicity of the roots. The method indicated above was implemented in [6] and employed in the present work.

The results of the numerical analysis also permit drawing the following conclusions. As the radius of the plate increases, the number of resonance curves increases. As the rigidity of the plate decreases (the parameter M increases) while its geometric dimensions remain fixed, the number of resonances also increases. The value of the first resonance frequency does not change with M (a and H are fixed) and corresponds to the resonance for an absolutely rigid stamp (Fig. 4, curves 1-3 correspond to $M = 4\cdot10^{-3}$, $4\cdot10^{-4}$, $4\cdot10^{-7}$, $p(r) = \delta(r - 0.1)$, $a = 7$, $R = 3.5$, $H = 0.8$, $\nu = 0.3$, $\nu_0 = 0.3$, $\gamma = 0.1$).

We note that the results of [5] for the limits of applicability of Winkler's hypothesis remain valid for the axisymmetric problem.

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